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Existence of positive solutions of some nonlinear elliptic problems in unbounded domains

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Abstract

In this paper, we study the existence of positive solutions of some nonlinear elliptic problems in unbounded domains. The existence is affected by the properties of the geometry and the topology of the domain.

Keywords: Palais–Smale sequence; The strip domain with a hole; Positive higher energy solution

1. Introduction

In this paper we are concerned with the following problem:

$$\begin{cases} -\Delta u + a(x)u = f(x,u) \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{cases} \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^N$ is an unbounded domain with smooth boundary $\partial\Omega$, $a(x)$ will be assumed throughout this paper locally Hölder continuous and satisfies

$$a_1 \geqslant a(x) \geqslant a_2 > 0, \quad \forall x \in \overline{\Omega}.$$

We will also assume that $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is of class C^1 and satisfies the following conditions:

(f1) $f(x, y) = o(y)$ near $y = 0$ uniformly in $x \in \overline{\Omega}$.
(f2) There exists $a_3 > 0$ such that $|f_y(x, y)| \leqslant a_3(1 + |y|^{p-1})$ $\forall x \in \overline{\Omega}$ and $y \in \mathbb{R}$, where $1 < p < \frac{N+2}{N-2}$ if $N > 2$ and $1 < p < \infty$ if $N = 1, 2$.
(f3) There exists $\theta > 2$ such that $0 < \theta F(x, y) \leqslant f(x, y)y$ $\forall x \in \overline{\Omega}$ and $y \in \mathbb{R}\backslash\{0\}$, where $F(x, y) = \int_0^y f(x, \tau)\, d\tau$.
(f4) $\frac{f(x,ty)y}{t}$ is an strictly increasing function of $t > 0$ $\forall x \in \overline{\Omega}$ and $y \in \mathbb{R}\backslash\{0\}$.

Since we seek only positive solutions of problem (1.1), it is convenient to define $f(x, u) \equiv 0$ for $u \leqslant 0$ and $x \in \overline{\Omega}$.

Associated to problem (1.1) is the energy functional I defined by

$$I(u) = \frac{1}{2} \int_\Omega \left(|\nabla u|^2 + a(x)u^2\right) dx - \int_\Omega F(x, u)\, dx,$$

which by above assumptions is well defined for $u \in H(\Omega)$ where

$$H(\Omega) = \left\{ u \in H_0^1(\Omega) \;\middle|\; \int_\Omega a(x)u^2\, dx < \infty \right\}.$$

$H(\Omega)$ become a Hilbert space, continuously embedded in $H_0^1(\Omega)$, when endowed with the inner product

$$\langle u, v\rangle_{H(\Omega)} = \int_\Omega \left(\nabla u \cdot \nabla v + a(x)uv\right) dx$$

whose associated norm we denote by $\|\cdot\|_{H(\Omega)}$. $I \in C^1(H(\Omega), \mathbb{R})$ (see [12]).

In the following definitions, we simply denote Palais–Smale by (PS).

Definition 1.1.

(1) *For $c \in \mathbb{R}$, a sequence $\{u_k\} \subset H(\Omega)$ is a $(PS)_c$-sequence if $I(u_k) \to c$ and $I'(u_k) \to 0$ in $H^{-1}(\Omega)$.*
(2) *$c \in \mathbb{R}$ is a (PS)-value if there exists a $(PS)_c$-sequence.*
(3) *I satisfies the $(PS)_c$-condition if every $(PS)_c$-sequence for I contains a convergent subsequence.*

It is well known that the solutions of problem (1.1) are the critical points of the energy functional I. Moreover, standard arguments from elliptic regularity theory show that critical points of I on $H(\Omega)$ are classical solutions of problem (1.1). However, the main difficulty in the study of problem (1.1) lies in the fact that Ω is unbounded; indeed, contrary to the case of bounded Ω, in this case the embedding of the Sobolev space $H(\Omega)$ into $L^p(\Omega)$ is not compact. Because of this lack of compactness, the standard variational methods do not apply (see [1]). In this connection we recall that for any bounded Ω there exists a solution of problem (1.1). In our case the result is no longer true: for instance, if Ω is a half space, in Esteban and Lions [8] it was proved that problem (1.1) admits no solution. The existence or the multiplicity of solutions is affected by the topology of the domain.

Denote that

$$M(\Omega)=\left\{u\in H(\Omega)\backslash\{0\}\;\middle|\;\int_{\Omega}\left(|\nabla u|^2+a(x)u^2\right)dx=\int_{\Omega}f(x,u)u\,dx\right\},$$
$$\alpha_M(\Omega)=\inf_{u\in M(\Omega)}I(u).$$

To look for solutions of problem (1.1) is also equivalent to find critical points of I constrained to lie upon the manifold $M(\Omega)$. As a consequence of Ekeland's variational principle, there exists a sequence $\{u_k\}\subset M(\Omega)$ such that

$$I(u_k)\to\alpha_M(\Omega),\qquad I'(u_k)\to 0\quad\text{in }H^{-1}(\Omega).$$

Although $\alpha_M(\Omega)$ does not guarantee that there exists a critical point $u\in H(\Omega)$ with $I(u)=\alpha_M(\Omega)$, we can analyze Palais–Smale sequences to justify if there exist positive solutions of problem (1.1). New analysis is needed to solve such problems which will be described as follows. Let

$$\Omega_k=\Omega\cap B_k^N(0),\quad\text{where }B_k^N(0)=\left\{x\in\mathbb{R}^N\;\middle|\;\|x\|<k\right\},$$
$$\tilde{\Omega}_k=\Omega\backslash\overline{B_k^N(0)}.$$

For $v\in H(\tilde{\Omega}_{k+1})$, it can be identified with an element of $H(\tilde{\Omega}_k)$ by extending v to be zero on $\tilde{\Omega}_k\backslash\tilde{\Omega}_{k+1}$.

In Section 2, let $\Theta(\Omega)$ be the set of all the positive (PS)-values. The set $\Theta(\Omega)$ in particular contains all the positive critical values of I. Let $\delta(\Omega)$ be the infimum of $\Theta(\Omega)$, it will be shown that $\Theta(\Omega)$ is a nonempty set, $\delta(\Omega)$ is a positive number, and the optimal lower bound for $\Theta(\Omega)$ is $\alpha_M(\Omega)$ when (f1)–(f4) are satisfied; that is to say, $\delta(\Omega)=\alpha_M(\Omega)$.

If u is a nontrivial solution of problem (1.1), multiplying the problem (1.1) by u and integrating by parts shows $u\in M(\Omega)$. For any $u\in H(\Omega)\backslash\{0\}$ and $t>0$, let $h_u(t)=I(tu)$. By (f1)–(f3), $h_u(0)=0$, $h_u(t)>0$ for t small, and $h_u(t)<0$ for t large. Therefore $\max_{t\geqslant 0}h_u(t)$ exists and is achieved at $t_u>0$, we get

$$h_u'(t_u)=0=t_u\|u\|^2_{H(\Omega)}-\int_{\Omega}f(x,t_uu)u\,dx$$

which implies $t_uu\in M(\Omega)$. Moreover by (f4), t_u is the unique value of $t>0$ such that $t_uu\in M(\Omega)$. This implies $M(\Omega)$ is radially homeomorphic to the unit ball in $H(\Omega)$.

In Section 3, we assert that if there exists a $(PS)_c$-sequence with $\alpha_M(\Omega)<c<\alpha_M(\tilde{\Omega}_k)$ for some large $k\in\mathbb{N}$, then there exist a ground state solution and a higher energy solution of problem (1.1). Next, if there exists a $(PS)_c$-sequence with $c>0$ and $c\notin\Theta(\tilde{\Omega}_m)$ for some $m\in\mathbb{N}$, then there exists a positive solution of problem (1.1).

In Section 4, as [6] and [7] we describe the (PS)-conditions, and give a necessary and sufficient in Ω in which I satisfies the $(PS)_{\alpha_M(\Omega)}$-condition.

In the last section, we study the domain is the strip domain with a hole, for simplicity in presentation, we consider the case where f and a do not depend on x, so the problem is as follows:

$$\begin{cases} -\Delta u + au = f(u) \quad \text{in } \Sigma, \\ u > 0 \quad \text{in } \Sigma, \\ u \in H_0^1(\Sigma), \end{cases} \tag{1.2}$$

where Σ is a strip domain with a hole. Denote that

$$A^r = \{(\xi, \eta) \in \mathbb{R}^{N-1} \times \mathbb{R} \mid |\xi| < r\}.$$

For the strip domain A^r, Chen [4] modified P.L. Lions [11] and Lien, Tzeng and Wang [10] to assert that there exists a ground state solution of problem (1.2) in A^r, and Chen, Chen and Wang [5] establish its asymptotic behavior and the solution is spherically symmetric in ξ and axially symmetric in η.

When $f(u) = u^p$, since Kwong [9] proved that problem (1.2) in $\mathbb{R}^N$ admits a unique solution, Benci and Cerami [2] asserted that problem (1.2) in exterior domains admits a higher energy solution. We use a new method different from Benci and Cerami [2] to prove there exists a positive higher energy solution of problem (1.2) in a strip domain with a hole.

2. The (*PS*)-value

We will introduce some preliminaries to analyze the behavior of Palais–Smale sequence and study the set $\Theta(\Omega)$ of all the positive (PS)-values.

Lemma 2.1. *If $\{u_k\}$ is a $(PS)_c$-sequence, then there exists a constant $\bar{c} > 0$ such that $\|u_k\|_{H(\Omega)} \leqslant \bar{c}$, and $c \geqslant 0$. If $c > 0$, then there exist a subsequence, still denoted by $\{u_k\}$, a constant $c' > 0$, such that $\|u_k\|_{H(\Omega)} \geqslant c'$.*

Proof. By (f3) and if k is large, then

$$\begin{aligned} c + o(1)\big(1 + \|u_k\|_{H(\Omega)}\big) &= I(u_k) - \frac{1}{\theta}\big(I'(u_k), u_k\big) \\ &= \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_k\|_{H(\Omega)}^2 - \int_\Omega \left[F(x, u_k) - \frac{1}{\theta} f(x, u_k)u_k\right] dx \\ &\geqslant \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_k\|_{H(\Omega)}^2. \end{aligned}$$

Thus $\|u_k\|_{H(\Omega)} \leqslant \bar{c}$. Then for large k, we have $(I'(u_k), u_k) = o(1)$ and

$$c + o(1) = I(u_k) - \frac{1}{\theta}\big(I'(u_k), u_k\big) \geqslant \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_k\|_{H(\Omega)}^2,$$

so $c \geqslant 0$.

Suppose that $c > 0$. If $\lim_{k\to\infty} \|u_k\|_{H(\Omega)} = 0$, then for large k, $I(u_k) = o(1)$, contradicts to $c > 0$. Thus there exist a subsequence, still denoted by $\{u_k\}$, a constant $c' > 0$, such that $\|u_k\|_{H(\Omega)} \geqslant c'$. $\square$

Lemma 2.2. *For any $u \in M(\Omega)$, there exists a constant $K > 0$ such that $I(u) \geqslant (\frac{\theta-2}{2\theta})(2K)^{\frac{-2}{p-1}} > 0$.*

Proof. By (f1) and (f2), for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$\left|f(x,u)\right| \leqslant \varepsilon|u| + C_\varepsilon|u|^p. \tag{2.1}$$

We take $\varepsilon = 1/2$, and by the Sobolev inequality,

$$\begin{aligned}
0 = \big(I'(u), u\big) &= \|u\|^2_{H(\Omega)} - \int_\Omega f(x,u)u\,dx \\
&\geqslant \|u\|^2_{H(\Omega)} - \int_\Omega \left(\frac{1}{2}u^2 + C_{1/2}|u|^{p+1}\right) dx \\
&\geqslant \frac{1}{2}\|u\|^2_{H(\Omega)} - K\|u\|^{p+1}_{H(\Omega)} = \|u\|^2_{H(\Omega)}\left(\frac{1}{2} - K\|u\|^{p-1}_{H(\Omega)}\right),
\end{aligned}$$

thus $\|u\|_{H(\Omega)} \geqslant (2K)^{\frac{-1}{p-1}}$, and then by (f3),

$$\begin{aligned}
I(u) &= \frac{1}{2}\|u\|^2_{H(\Omega)} - \int_\Omega F(x,u)\,dx \geqslant \frac{1}{2}\|u\|^2_{H(\Omega)} - \frac{1}{\theta}\int_\Omega f(x,u)u\,dx \\
&= \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u\|^2_{H(\Omega)} \geqslant \frac{\theta-2}{2\theta}(2K)^{\frac{-2}{p-1}}. \qquad \square
\end{aligned}$$

Notice that $\delta(\Omega)$, the infimum of all the positive (PS)-values, is a positive number proved as follows. By Stuart [13], $\alpha_M(\Omega)$ is a positive $(PS)_{\alpha_M(\Omega)}$-value, so $\Theta(\Omega)$ is not empty and $\delta(\Omega) \leqslant \alpha_M(\Omega)$.

Lemma 2.3. *If* (f1)–(f3) *hold, then $\alpha_M(\Omega) \geqslant \delta(\Omega) > 0$.*

Proof. Obviously, by the definition of $\Theta(\Omega)$, $\alpha_M(\Omega) \geqslant \delta(\Omega)$. Let $\{u_k\}$ be a $(PS)_c$-sequence with $c > 0$, by Lemma 2.1, $\{u_k\}$ is bounded and there exist a subsequence, still denoted by $\{u_k\}$, a constant $c' > 0$, such that $\|u_k\|_{H(\Omega)} \geqslant c'$. Since $\|u_k\|^2_{H(\Omega)} = \int_\Omega f(x,u_k)u_k\,dx + o(1)$ for large k, then by (f3), if k is large, we have

$$\begin{aligned}
c + o(1) = I(u_k) &= \frac{1}{2}\|u_k\|^2_{H(\Omega)} - \int_\Omega F(x,u_k)\,dx \\
&\geqslant \frac{1}{2}\|u_k\|^2_{H(\Omega)} - \frac{1}{\theta}\int_\Omega f(x,u_k)u_k\,dx
\end{aligned}$$

$$= \left(\frac{1}{2} - \frac{1}{\theta}\right) \|u_k\|^2_{H(\Omega)} + o(1)$$
$$\geqslant \left(\frac{1}{2} - \frac{1}{\theta}\right) c' + o(1).$$

Since c is arbitrary positive (PS)-value, hence $\delta(\Omega) \geqslant (\frac{1}{2} - \frac{1}{\theta})c' > 0$. $\square$

We introduce for an arbitrary sequence $\{u_k\}$ bounded in $L^2(\mathbb{R}^N)$ the concentration functions of $|u_k|^2$,

$$\Phi_k(t) = \sup_{z \in \mathbb{R}^N} \int_{B_t^N(z)} |u_k|^2,$$

defined for $t \geqslant 0$.

Lemma 2.4. *Let $\{u_k\}$ be bounded in $H(\mathbb{R}^N)$ and assume that for some $t_0 > 0$,*

$$\Phi_k(t_0) \to 0.$$

Then

$$u_k \to 0 \quad \textit{strongly in } L^q\big(\mathbb{R}^N\big) \quad \textit{for all } 2 < q < 2^* = \frac{2N}{N-2}.$$

If in addition u_k satisfies $(I'(u_k), u_k) \to 0$, then

$$u_k \to 0 \quad \textit{strongly in } H\big(\mathbb{R}^N\big).$$

Proof. We divide the proof into several steps:

(1) Decompose $\mathbb{R}^N$ into unit cubes $F_0 = \{P_i^1\}_{i=1}^\infty$ of length 1 with vertex at lattice points. Continuing to bisect the cubes to obtain cubes $F_m = \{P_i^m\}_{i=1}^\infty$ of length of each P_i^m is $\frac{1}{2^m}$. Let m_0 satisfies $\sqrt{N}\frac{1}{2^{m_0}} < t_0$. For each i, let $B_i^{m_0}$ be a ball with center at the same as that of $P_i^{m_0}$ and of radius t_0 in $\mathbb{R}^N$. Then $P_i^{m_0} \subset B_i^{m_0}$, $\mathbb{R}^N = \bigcup_{i=1}^\infty P_i^{m_0}$, and $\{P_i^{m_0}\}_{i=1}^\infty$ are nonoverlapping. Write $P_i = P_i^{m_0}$. If we take q and r such that $2 < q < r < 2^*$, we can write, by the Hölder inequality and Sobolev imbedding,

$$\int_{\mathbb{R}^N} |u_k|^q = \sum_{i=1}^\infty \int_{P_i} |u_k|^q$$
$$= \sum_{i=1}^\infty \int_{P_i} |u_k|^{2\frac{r-q}{r-2}} |u_k|^{r\frac{q-2}{r-2}}$$
$$\leqslant \sum_{i=1}^\infty \left(\int_{P_i} |u_k|^2\right)^{\frac{r-q}{r-2}} \left(\int_{P_i} |u_k|^r\right)^{\frac{q-2}{r-2}}$$

$$\leqslant \left(\Phi_k(t_0)\right)^{\frac{r-q}{r-2}} \sum_{i=1}^{\infty} \left(\int_{P_i} |u_k|^r \right)^{\frac{q-2}{r-2}}$$

$$\leqslant c\left(\Phi_k(t_0)\right)^{\frac{r-q}{r-2}} \sum_{i=1}^{\infty} \left(\int_{P_i} \left(|\nabla u_k|^2 + a(x)u_k^2\right) \right)^{\frac{r}{2}\frac{q-2}{r-2}}.$$

Since $\lim_{r\to q} \frac{r}{2}\frac{q-2}{r-2} = \frac{q}{2} > 1$, we may choose r such that $s = \frac{r}{2}\frac{q-2}{r-2} \geqslant 1$,

$$\sum_{i=1}^{\infty} \left(\int_{P_i} \left(|\nabla u_k|^2 + a(x)|u_k|^2\right) \right)^{\frac{r}{2}\frac{q-2}{r-2}}$$

$$= \sum_{i=1}^{\infty} \left(\int_{P_i} \left(|\nabla u_k|^2 + a(x)|u_k|^2\right) \right)^{s} \leqslant \left(\sum_{i=1}^{\infty} \int_{P_i} \left(|\nabla u_k|^2 + a(x)|u_k|^2\right) \right)^{s}$$

$$= \left(\int_{\mathbb{R}^N} \left(|\nabla u_k|^2 + a(x)|u_k|^2\right) \right)^{s} = \|u_k\|_{H(\mathbb{R}^N)}^{2s} \leqslant c.$$

Therefore

$$u_k \to 0 \text{ strongly in } L^q\left(\mathbb{R}^N\right) \quad \text{for all } 2 < q < 2^* = \frac{2N}{N-2}.$$

(2) If in addition u_k satisfies $(I'(u_k), u_k) \to 0$, then for large k, $\|u_k\|_{H(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} f(x, u_k)u_k\,dx + o(1)$. By (2.1),

$$\|u_k\|_{H(\mathbb{R}^N)}^2 \leqslant \int_{\mathbb{R}^N} \left|f(x, u_k)\right||u_k|\,dx + o(1)$$

$$\leqslant \varepsilon\|u_k\|_{H(\mathbb{R}^N)}^2 + C_\varepsilon\|u_k\|_{L^{p+1}(\mathbb{R}^N)}^{p+1} + o(1),$$

or by part (1),

$$(1-\varepsilon)\|u_k\|_{H(\mathbb{R}^N)}^2 \leqslant C_\varepsilon\|u_k\|_{L^{p+1}(\mathbb{R}^N)}^{p+1} + o(1) = o(1).$$

Hence

$$u_k \to 0 \text{ strongly in } H\left(\mathbb{R}^N\right). \qquad \square$$

Lemma 2.5. *Let $\{u_k\}$ be a $(PS)_c$-sequence with $c > 0$. Then there exists a sequence $\{t_k\}$ in $\mathbb{R}_+$ such that $\{t_k u_k\} \subset M(\Omega)$, $\{t_k\}$ is bounded, and $\alpha_M(\Omega) \leqslant I(t_k u_k) \leqslant c + o(1)$.*

Proof. Let $\{u_k\}$ be a $(PS)_c$-sequence with $c > 0$, thus for large k, $u_k \nleqslant 0$, and

$$u_k \nrightarrow 0 \text{ strongly in } H\left(\mathbb{R}^N\right),$$

where u_k is identified with an element of $H(\mathbb{R}^N)$ by extending u_k to be zero on $\mathbb{R}^N\backslash\Omega$, then by Lemma 2.4, there exist a sequence $\{z_k\}\subset\mathbb{R}^N$ and $\varepsilon_1>0$ such that $u_k\nleqslant 0$ in $B^N_{1/2}(z_k)$, and

$$\int\limits_{B^N_{1/2}(z_k)} \left|u_k(x)\right|^2 dx \geqslant \varepsilon_1.$$

Hence there exist $\varepsilon_2>0$, $\varepsilon_3>0$, such that

$$|D_k|\equiv\left|\left\{x\in B^N_{1/2}(z_k)\;\middle|\; u_k(x)\geqslant\varepsilon_2\right\}\right|\geqslant\varepsilon_3,$$

where $|D_k|$ is the Lebesgue measure of the set D_k.

For $u_k\not\equiv 0$, by (f4), there exists a unique positive number t_k such that $t_k u_k\in M(\Omega)$, then

$$\|u_k\|^2_{H(\Omega)}=\int\limits_{\Omega}\frac{f(x,t_k u_k)u_k}{t_k}\,dx.$$

Either $t_k\leqslant 1$, or $t_k>1$ in which case by (f3), $\frac{F(x,ty)}{t^\theta}$ is an nondecreasing function of $t>0$ for all $x\in\Omega$ and $y\in\mathbb{R}\backslash\{0\}$, then

$$t_k^2\|u_k\|^2_{H(\Omega)}=\int\limits_{\Omega} f(x,t_k u_k)t_k u_k\,dx\geqslant\theta\int\limits_{\Omega}F(x,t_k u_k)\,dx\geqslant\theta\int\limits_{\Omega}t_k^\theta F(x,u_k)\,dx.$$

Consequently by Lemma 2.1,

$$\begin{aligned} t_k^{\theta-2}&\leqslant\theta^{-1}\frac{\|u_k\|^2_{H(\Omega)}}{\int_\Omega F(x,u_k)\,dx}\leqslant\theta^{-1}\frac{\bar{c}^2}{\int_{D_k}F(x,u_k)\,dx}\leqslant\theta^{-1}\frac{\bar{c}^2}{\int_{D_k}F(x,\varepsilon_2)\,dx}\\ &\leqslant\theta^{-1}\frac{\bar{c}^2}{\varepsilon_3(\min_{x\in\overline{D_k}}F(x,\varepsilon_2))},\end{aligned}$$

thus $\{t_k\}$ must be bounded.

$$\begin{aligned} &I(t_k u_k)-I(u_k)\\ &\quad=\frac{1}{2}t_k^2\|u_k\|^2_{H(\Omega)}-\int\limits_{\Omega}F(x,t_k u_k)\,dx-\frac{1}{2}\|u_k\|^2_{H(\Omega)}+\int\limits_{\Omega}F(x,u_k)\,dx\\ &\quad=\frac{1}{2}\left(t_k^2-1\right)\int\limits_{\Omega}f(x,u_k)u_k\,dx-\int\limits_{\Omega}F(x,t_k u_k)\,dx+\int\limits_{\Omega}F(x,u_k)\,dx+o(1)\\ &\quad=g(t_k)+o(1),\end{aligned}$$

where $g(t)=\frac{1}{2}(t^2-1)\int_\Omega f(x,u_k)u_k\,dx-\int_\Omega F(x,tu_k)\,dx+\int_\Omega F(x,u_k)\,dx$. Since

$$\begin{aligned} g'(t)&=t\int\limits_{\Omega}f(x,u_k)u_k\,dx-\int\limits_{\Omega}f(x,tu_k)u_k\,dx\\ &=t\left(\int\limits_{\Omega}\frac{f(x,u_k)u_k}{1}\,dx-\int\limits_{\Omega}\frac{f(x,tu_k)u_k}{t}\,dx\right),\end{aligned}$$

it follows from (f4) that $g'(t) > 0$ if $t \in (0,1)$ and $g'(t) < 0$ if $t \in (1,\infty)$. Thus $g(1) = \max_{t\in[0,\infty)} g(t)$, $g(1) = 0$, and for large k,

$$I(t_k u_k) - I(u_k) = g(t_k) + o(1) \leqslant g(1) + o(1) = o(1).$$

Hence $\alpha_M(\Omega) \leqslant I(t_k u_k) \leqslant c + o(1)$. □

Next, we prove that an optimal lower bound for $\Theta(\Omega)$ is $\alpha_M(\Omega)$ when (f1)–(f4) are satisfied.

Theorem 2.6. *If* (f1)–(f4) *hold, then* $\delta(\Omega) = \alpha_M(\Omega)$.

Proof. It suffices to show $\delta(\Omega) \geqslant \alpha_M(\Omega)$, since the reversed inequality is always true. Let $\{u_k\}$ be a $(PS)_c$-sequence with $c > 0$, and by Lemma 2.5, there exists $t_k \in (0,\infty)$, such that $t_k u_k \in M(\Omega)$, $\{t_k\}$ is bounded, and $\alpha_M(\Omega) \leqslant I(t_k u_k) \leqslant c + o(1)$. Since c is arbitrary positive (PS)-value, it follows that $\alpha_M(\Omega) \leqslant \delta(\Omega)$. □

3. Existence of solutions

In this section, we assert that if there exists a $(PS)_c$-sequence with $\alpha_M(\Omega) < c < \alpha_M(\tilde{\Omega}_k)$ for some large $k \in \mathbb{N}$, then there exist at least two positive solutions of problem (1.1); i.e., a ground state solution and a positive higher energy solution. Next, if there exists a $(PS)_c$-sequence with $c > 0$ and $c \notin \Theta(\tilde{\Omega}_m)$ for some $m \in \mathbb{N}$, then there exists a positive higher energy solution of problem (1.1).

Lemma 3.1. *Let* $\{u_k\}$ *be a* (PS)*-sequence for* I *satisfying* $u_k \rightharpoonup u$ *weakly in* $H(\Omega)$*. Then*

(1) *u is a weak solution of problem* (1.1).
(2) *If $u \not\equiv 0$, then u is a positive solution of problem* (1.1).
(3) *If $\{u_k\}$ is a $(PS)_{\alpha_M(\Omega)}$-sequence for I satisfying $u_k \rightharpoonup u$ weakly in $H(\Omega)$ and $u \not\equiv 0$, then $u_k \to u$ strongly in $H(\Omega)$.*

Proof. (1) Take a subsequence $\{u_k\}$ such that $u_k \rightharpoonup u$ weakly in $H(\Omega)$, a.e. in Ω, and strongly in $L^q_{\mathrm{loc}}(\Omega)$ where $1 \leqslant q < 2^*$. For $\phi \in C_c^\infty(\Omega)$, we get

$$\int_\Omega \nabla u_k \cdot \nabla\phi \to \int_\Omega \nabla u \cdot \nabla\phi,$$

$$\int_\Omega a(x) u_k \phi \to \int_\Omega a(x) u \phi,$$

and by (f2), $|f(x,u_k) - f(x,u)||\phi| \leqslant a_3(|u_k| + |u_k|^p + |u| + |u|^p)|\phi|$, then by the generalization of the Lebesgue dominated convergence theorem, we have

$$\int_\Omega f(x,u_k)\phi \to \int_\Omega f(x,u)\phi.$$

Hence

$$\big(I'(u), \phi\big) = \lim_{k\to\infty} \big(I'(u_k), \phi\big) = 0.$$

Since $C_c^\infty(\Omega)$ is dense in $H(\Omega)$, we have $I'(u) = 0$. Therefore u is a weak solution of problem (1.1).

(2) If u is a nonzero solution of problem (1.1), then $u \in M(\Omega)$. By elliptic regularity, any critical point of I is a classical solution of problem (1.1). Let $u^-(x) = \max(-u(x), 0)$. Since

$$0 = \big(I'(u), u^-\big) = \int_\Omega \nabla u \cdot \nabla u^- + \int_\Omega a(x)uu^- - \int_\Omega f(x,u)u^- = -\|u^-\|^2_{H(\Omega)},$$

hence $u \geqslant 0$. By the maximum principle, $u > 0$ in Ω.

(3) By part (2), $u \in M(\Omega)$ and applying Fatou's lemma yields

$$\begin{aligned}\alpha_M(\Omega) \leqslant I(u) &= \frac{1}{2}\|u\|^2_{H(\Omega)} - \int_\Omega F(x,u)\,dx = \frac{1}{2}\int_\Omega f(x,u)u\,dx - \int_\Omega F(x,u)\,dx \\ &\leqslant \liminf_{k\to\infty} \int_\Omega \left(\frac{1}{2} f(x,u_k)u_k - F(x,u_k)\right) dx = \lim_{k\to\infty} I(u_k) = \alpha_M(\Omega),\end{aligned}$$

or

$$I(u) = \alpha_M(\Omega). \tag{3.1}$$

Set $p_k = u_k - u$ to get $p_k \rightharpoonup 0$ weakly in $H(\Omega)$, a.e. in Ω, and strongly in $L^q_{\mathrm{loc}}(\Omega)$ where $1 \leqslant q < 2^*$, then if k is large, we have

$$\|p_k\|^2_{H(\Omega)} = \|u_k\|^2_{H(\Omega)} - \|u\|^2_{H(\Omega)} + o(1). \tag{3.2}$$

Since $u \in H(\Omega)$, by (f1) and (f2), for any $\varepsilon > 0$, there exists $r > 0$ such that for $\tilde{\Omega}_r = \Omega \backslash \overline{B_r^N(0)}$,

$$\int_{\tilde{\Omega}_r} |u|^2 < \varepsilon, \qquad \int_{\tilde{\Omega}_r} |u|^{p+1} < \varepsilon, \qquad \int_{\tilde{\Omega}_r} \big|F(x,u)\big| < \varepsilon. \tag{3.3}$$

By the generalization of the Lebesgue dominated convergence theorem, we have

$$\int_{\Omega_r} F(x,u_k) \to \int_{\Omega_r} F(x,u) \quad \text{and} \quad \int_{\Omega_r} F(x,p_k) \to 0.$$

Then

$$\int_{\Omega_r} \big|F(x,p_k) - F(x,u_k) + F(x,u)\big| < \varepsilon. \tag{3.4}$$

Now by the Hölder inequality and $\|p_k\|_{H(\Omega)}$ and $\|u_k\|_{H(\Omega)}$ are bounded.

$$\begin{aligned}
&\int_{\tilde{\Omega}_r} \bigl|F(x,p_k)-F(x,u_k)\bigr| \\
&\quad = \int_{\tilde{\Omega}_r} \bigl|f\bigl(x,tp_k+(1-t)u_k\bigr)\bigr||u| \quad \text{for some } 0<t<1 \\
&\quad \leqslant c\int_{\tilde{\Omega}_r} \bigl(|p_k|+|u_k|+|p_k|^p+|u_k|^p\bigr)|u| \\
&\quad \leqslant c\bigl(\|p_k\|_{H(\Omega)}\|u\|_{L^2(\tilde{\Omega}_r)}+\|u_k\|_{H(\Omega)}\|u\|_{L^2(\tilde{\Omega}_r)} \\
&\qquad + \|p_k\|_{H(\Omega)}^p\|u\|_{L^{p+1}(\tilde{\Omega}_r)}+\|u_k\|_{H(\Omega)}^p\|u\|_{L^{p+1}(\tilde{\Omega}_r)}\bigr) \\
&\quad < c\varepsilon. \qquad (3.5)
\end{aligned}$$

Therefore by (3.3)–(3.5), if k is large,

$$\int_{\Omega} F(x,p_k)=\int_{\Omega} F(x,u_k)-\int_{\Omega} F(x,u)+o(1). \qquad (3.6)$$

By (3.1), (3.2), and (3.6), if k is large,

$$I(p_k)=I(u_k)-I(u)+o(1)=\alpha_M(\Omega)-\alpha_M(\Omega)+o(1)=o(1),$$

and it follows that

$$\|p_k\|_{H(\Omega)}^2=2\int_{\Omega} F(x,p_k)\,dx. \qquad (3.7)$$

For $\phi\in C_c^\infty(\Omega)$, by (f2), we have $|f(x,u_k-u)\phi|\leqslant a_3(|u_k-u|+|u_k-u|^p)|\phi|$, then by the generalization of the Lebesgue dominated convergence theorem, we have $\int_\Omega f(x,u_k-u)\phi\to 0$. Since $C_c^\infty(\Omega)$ is dense in $H(\Omega)$, $f(x,u_k-u)\to 0$ in $H^{-1}(\Omega)$. Similarly, $f(x,u_k)-f(x,u)\to 0$ in $H^{-1}(\Omega)$. So for large k, we have

$$\begin{aligned}
I'(p_k) &= -\Delta p_k+a(x)p_k-f(x,p_k) \\
&= -\Delta(u_k-u)+a(x)(u_k-u)-f(x,u_k-u) \\
&= \bigl(-\Delta u_k+a(x)u_k-f(x,u_k)\bigr)-\bigl(-\Delta u+a(x)u-f(x,u)\bigr) \\
&\quad -\bigl(f(x,u_k-u)-f(x,u_k)+f(x,u)\bigr) \\
&= I'(u_k)-I'(u)+o(1)=o(1),
\end{aligned}$$

and it follows that

$$\|p_k\|_{H(\Omega)}^2=\int_{\Omega} f(x,p_k)p_k\,dx. \qquad (3.8)$$

From (f3), (3.7), and (3.8), we have $\|p_k\|_{H(\Omega)}=o(1)$ for large k; that is to say, $u_k\to u$ strongly in $H(\Omega)$. $\square$

We shall see what will happen whenever u is zero. Let

$$\Omega_k=\Omega\cap B_k^N(0); \qquad \tilde{\Omega}_k=\Omega\backslash\overline{B_k^N(0)}.$$

Lemma 3.2. *Let $\{u_k\}$ be a $(PS)_c$-sequence with $c > 0$. Then*

(1) *If* (f1)–(f3) *hold, suppose that $u_k \rightharpoonup 0$ weakly in $H(\Omega)$, then for each $1 \leqslant q < 2^*$, there exists a subsequence $\{u_k\}$ such that for Ω_{2k}, if k is large,*

$$\int_{\Omega_{2k}} |u_k|^q = o(1).$$

(2) *In addition to* (f1)–(f3)*, that* (f4) *satisfied, suppose for each $1 \leqslant q < 2^*$, there exists a subsequence $\{u_k\}$ such that for Ω_{2k}, if k is large,*

$$\int_{\Omega_{2k}} |u_k|^q = o(1).$$

Then we have $c \geqslant \alpha_M(\tilde{\Omega}_k)$ for all large k.

(3) *If* (f1)–(f4) *hold, suppose that $u_k \rightharpoonup 0$ weakly in $H(\Omega)$, then $c \geqslant \alpha_M(\tilde{\Omega}_k)$ for all large k.*

Proof. (1) Since $u_k \rightharpoonup 0$ weakly in $H(\Omega)$, a.e. in Ω, and strongly in $L^q_{\mathrm{loc}}(\Omega)$, where $1 \leqslant q < 2^*$. Thus for each $m \in \mathbb{N}$, $\lim_{k\to\infty} \int_{\Omega_m} |u_k|^q = 0$. We can take a subsequence $\{u_{k_m}\}$ such that $\int_{\Omega_m} |u_{k_m}|^q < 1/m$. Therefore for each $1 \leqslant q < 2^*$, there exists a subsequence, still denoted by $\{u_k\}$ such that for Ω_{2k}, for large k, $\int_{\Omega_{2k}} |u_k|^q = o(1)$.

(2) Let $\{u_k\}$ be a $(PS)_c$-sequence, so for large k,

$$I(u_k) = \frac{1}{2}\|u_k\|^2_{H(\Omega)} - \int_\Omega F(x, u_k)\,dx = c + o(1),$$

$$\|u_k\|^2_{H(\Omega)} = \int_\Omega f(x, u_k)u_k\,dx + o(1).$$

Let $\zeta \in C^\infty([0, \infty))$ such that

$$0 \leqslant \zeta \leqslant 1, \quad \zeta(t) = \begin{cases} 0 & \text{for } t \in [0, 1], \\ 1 & \text{for } t \in [2, \infty). \end{cases}$$

Let $\zeta_k(x) = \zeta(|x|/k)$. Since $\{\zeta_k^2 u_k\}$ is bounded in $H(\Omega)$, if k is large,

$$\begin{aligned} o(1) &= \left(I'(u_k), \zeta_k^2 u_k\right) \\ &= \int_\Omega \left(\zeta_k^2|\nabla u_k|^2 + 2\zeta_k u_k \nabla\zeta_k \cdot \nabla u_k + a(x)\zeta_k^2 u_k^2\right) - \int_\Omega f(x, u_k)\zeta_k^2 u_k. \end{aligned} \tag{3.9}$$

Note that $|\nabla\zeta_k(x)| \leqslant c/k$, for each $1 \leqslant q < 2^*$, if k is large, $\int_{\Omega_{2k}} |u_k|^q = o(1)$, so for large k,

$$\int_\Omega \zeta_k u_k \nabla\zeta_k \cdot \nabla u_k = o(1), \tag{3.10}$$

and by (f2),

$$\int_{\Omega_{2k}} f(x,u_k)u_k\,dx = o(1),$$

then we have

$$\int_{\Omega} f(x,u_k)\zeta_k^2 u_k\,dx = \int_{\Omega} f(x,u_k)u_k\,dx + o(1) = \|u_k\|^2_{H(\Omega)} + o(1), \tag{3.11}$$

by (f2) again,

$$\int_{\Omega} f(x,\zeta_k u_k)\zeta_k u_k\,dx = \int_{\Omega} f(x,u_k)\zeta_k u_k\,dx + o(1) = \|u_k\|^2_{H(\Omega)} + o(1), \tag{3.12}$$

and

$$\begin{aligned}\int_{\Omega} F(x,\zeta_k u_k)\,dx &= \int_{\Omega} \big(F(x,\zeta_k u_k) - F(x,u_k)\big)\,dx + \int_{\Omega} F(x,u_k)\,dx\\ &= \int_{\Omega_{2k}} f\big(x,(1-t)u_k + t\zeta_k u_k\big)(\zeta_k u_k - u_k)\,dx + \int_{\Omega} F(x,u_k)\,dx\\ &= \int_{\Omega} F(x,u_k)\,dx + o(1) \quad \text{for some } 0<t<1. \end{aligned}\tag{3.13}$$

For large k, substituting (3.10), (3.11), into (3.9) yields

$$\int_{\Omega} \zeta_k^2\big(|\nabla u_k|^2 + a(x)u_k^2\big) = \|u_k\|^2_{H(\Omega)} + o(1). \tag{3.14}$$

Then by (3.13) and (3.14), for large k,

$$\begin{aligned} I(\zeta_k u_k) &= \frac{1}{2}\int_{\Omega} \big[|\nabla\zeta_k|^2 u_k^2 + \zeta_k^2\big(|\nabla u_k|^2 + a(x)u_k^2\big) + 2\zeta_k u_k \nabla\zeta_k\cdot\nabla u_k\big]\\ &\quad - \int_{\Omega} F(x,\zeta_k u_k)\,dx\\ &= \frac{1}{2}\|u_k\|^2_{H(\Omega)} - \int_{\Omega} F(x,u_k)\,dx + o(1)\\ &= I(u_k) + o(1) = c + o(1). \end{aligned}\tag{3.15}$$

By (3.14), (3.12), for large k,

$$\begin{aligned}\big(I'(\zeta_k u_k),\zeta_k u_k\big) &= \|\zeta_k u_k\|^2_{H(\Omega)} - \int_{\Omega} f(x,\zeta_k u_k)\zeta_k u_k\,dx\\ &= \|u_k\|^2_{H(\Omega)} - \|u_k\|^2_{H(\Omega)} + o(1) = o(1). \end{aligned}\tag{3.16}$$

Let $v_k = \zeta_k u_k \in H(\tilde{\Omega}_k)$. For $v_k \not\equiv 0$, by (3.15), (3.16) and Lemma 2.5, there exists $t_k \in (0, \infty)$ such that $t_k v_k \in M(\tilde{\Omega}_k)$, $\{t_k\}$ is bounded, and for large k, $\alpha_M(\tilde{\Omega}_k) \leqslant I(t_k v_k) \leqslant I(v_k) + o(1) = c + o(1)$. So we have $\alpha_M(\tilde{\Omega}_k) \leqslant c$ for large k.

(3) By part (1) and part (2). □

Now we will prove the existence of positive solutions of problem (1.1).

Theorem 3.3. *Suppose* (f1)–(f4) *hold, there exists a ground state solution u of problem* (1.1) *with $I(u) = \alpha_M(\Omega)$ if $\alpha_M(\Omega) < \alpha_M(\tilde{\Omega}_k)$ for some large $k \in \mathbb{N}$.*

Proof. As a consequence of Ekeland's variational principle, there exists a sequence $\{u_k\} \subset M(\Omega)$ which weakly converges to u , such that $\{u_k\}$ is a $(PS)_{\alpha_M(\Omega)}$-sequence. If $\alpha_M(\Omega) < \alpha_M(\tilde{\Omega}_k)$ for some large $k \in \mathbb{N}$, by Lemma 3.2, replacing c by $\alpha_M(\Omega)$, $u \not\equiv 0$, and then by Lemma 3.1, $u > 0$, $u_k \to u$ strongly in $H(\Omega)$, and $I(u) = \alpha_M(\Omega)$. □

Theorem 3.4. *If* (f1)–(f4) *hold, suppose there exists a $(PS)_c$-sequence with $\alpha_M(\Omega) < c < \alpha_M(\tilde{\Omega}_k)$ for some large $k \in \mathbb{N}$, then there exists a higher energy solution v of problem* (1.1) *with $c \geqslant I(v) > \alpha_M(\Omega)$.*

Proof. Let $\{v_k\} \subset H(\Omega)$ be a $(PS)_c$-sequence with $\alpha_M(\Omega) < c < \alpha_M(\tilde{\Omega}_k)$ for some large $k \in \mathbb{N}$. Take a subsequence $\{v_k\}$ such that $v_k \rightharpoonup v$ weakly in $H(\Omega)$, a.e. in Ω, and strongly in $L^q_{\mathrm{loc}}(\Omega)$ where $1 \leqslant q < 2^*$. By Lemma 3.2, $v \not\equiv 0$, then by Lemma 3.1, v is a positive solution of problem (1.1) with $c \geqslant I(v) \geqslant \alpha_M(\Omega)$.

Suppose $I(v) = \alpha_M(\Omega)$. From Theorem 3.3, $I(u) = \alpha_M(\Omega)$. Setting $w_k = v_k - v$, as the same line of the proof of Lemma 3.1(3), for large k,

$$I(w_k) = I(v_k) - I(v) + o(1) = c - \alpha_M(\Omega) + o(1),$$
$$I'(w_k) = I'(v_k) - I'(v) + o(1) = o(1),$$

so $\{w_k\}$ is a $(PS)_{c-\alpha_M(\Omega)}$-sequence. Since $0 < c - \alpha_M(\Omega) < \alpha_M(\tilde{\Omega}_k)$ for some large $k \in \mathbb{N}$, by above arguments it follows that $w_k \rightharpoonup w \not\equiv 0$, a contradiction. □

Theorem 3.5. *If* (f1)–(f4) *hold, suppose there exists a $(PS)_c$-sequence with $c > 0$ and $c \notin \Theta(\tilde{\Omega}_m)$ for some $m \in \mathbb{N}$, then there exists a positive solution of problem* (1.1).

Proof. Let $\{u_k\}$ be a $(PS)_c$-sequence with $c > 0$ and $c \notin \Theta(\tilde{\Omega}_m)$ for some $m \in \mathbb{N}$. Take a subsequence $\{u_k\}$ such that $u_k \rightharpoonup u$ weakly in $H(\Omega)$, a.e. in Ω, and strongly in $L^q_{\mathrm{loc}}(\Omega)$ where $1 \leqslant q < 2^*$. Moreover, $I'(u) = 0$ and $I(u) \leqslant c$. We claim that $u \not\equiv 0$. Suppose $u \equiv 0$, following the proof of Lemma 3.2, for each $1 \leqslant q < 2^*$, there exists a subsequence $\{u_k\}$ such that for Ω_k,

$$\int_{\Omega_k} |u_k|^q = o(1). \tag{3.17}$$

Let $\xi:\mathbb{R}^N\to[0,1]$ be a C^∞-function which satisfies

$$\xi(x)=\begin{cases}0 & \text{for } x\in B_m^N(0),\\ 1 & \text{for } x\notin B_{m+1}^N(0).\end{cases}$$

Let $w_k=\xi u_k$, $w_k\in H(\tilde{\Omega}_m)$. Then we want to show that $\{w_k\}$ is a $(PS)_c$-sequence in $H(\tilde{\Omega}_m)$.

It suffices to show that

$$\lim_{k\to\infty}\bigl|I(w_k)-I(u_k)\bigr|=0 \tag{3.18}$$

and

$$\lim_{k\to\infty}\sup_{\|\phi\|_{H^1(\tilde{\Omega}_m)}\leqslant 1}\bigl|\bigl(I'(w_k),\phi\bigr)-\bigl(I'(u_k),\phi\bigr)\bigr|=0. \tag{3.19}$$

By a direct computation,

$$\begin{aligned}
&\bigl|\bigl(I'(w_k),\phi\bigr)-\bigl(I'(u_k),\phi\bigr)\bigr|\\
&\quad\leqslant\Bigl|\int_{\tilde{\Omega}_m}a(x)\bigl(\xi(x)-1\bigr)u_k\phi\Bigr|+\Bigl|\int_{\tilde{\Omega}_m}\bigl(\xi(x)-1\bigr)\nabla u_k\cdot\nabla\phi\Bigr|\\
&\qquad+\Bigl|\int_{\tilde{\Omega}_m}u_k\nabla\xi\cdot\nabla\phi\Bigr|+\Bigl|\int_{\tilde{\Omega}_m}\bigl(f(u_k)-f(\xi u_k)\bigr)\phi\Bigr|\\
&\quad\leqslant a_1\Bigl(\int_{\tilde{\Omega}_m\cap B_{m+1}^N(0)}|u_k|^2\Bigr)^{\frac12}+\Bigl(\int_{\tilde{\Omega}_m\cap B_{m+1}^N(0)}|\nabla u_k|^2\Bigr)^{\frac12}\\
&\qquad+\|\nabla\xi\|_{L^\infty}\Bigl(\int_{\tilde{\Omega}_m\cap B_{m+1}^N(0)}|u_k|^2\Bigr)^{\frac12}\\
&\qquad+2a_3\Bigl[\Bigl(\int_{\tilde{\Omega}_m\cap B_{m+1}^N(0)}|u_k|^2\Bigr)^{\frac12}+\Bigl(\int_{\tilde{\Omega}_m\cap B_{m+1}^N(0)}|u_k|^{2p}\Bigr)^{\frac12}\Bigr].
\end{aligned}\tag{3.20}$$

Since $\tilde{\Omega}_m\cap B_{m+1}^N(0)\subset\Omega_{k-1}$ if k is large, (3.19) follows from (3.20) and (3.17), provided that

$$\int_{\tilde{\Omega}_m\cap B_{m+1}^N(0)}|\nabla u_k|^2\to 0\quad\text{as } k\to\infty. \tag{3.21}$$

Now we prove (3.21). Let $\xi_k:\mathbb{R}^N\to[0,1]$ be a C_0^∞-function which satisfies $0\leqslant\xi_k\leqslant 1$, $|\nabla\xi_k|\leqslant 1$, and

$$\xi_k(x)=\begin{cases}1 & \text{for } x\in B_{k-1}^N(0),\\ 0 & \text{for } x\notin B_k^N(0).\end{cases}$$

Since $\{\xi_k u_k\}$ is bounded in $H(\Omega)$, if k is large,

$$\begin{aligned} o(1) &= \big(I'(u_k), \xi_k u_k\big) \\ &= \int\limits_{\Omega_k} \xi_k |\nabla u_k|^2 + \int\limits_{\Omega_k} u_k \nabla \xi_k \cdot \nabla u_k + \int\limits_{\Omega_k} a(x)\xi_k u_k^2 - \int\limits_{\Omega_k} f(x, u_k)\xi_k u_k. \end{aligned} \tag{3.22}$$

By (3.17), we conclude that the last three integrals of (3.22) tend to zero as $k \to \infty$ and consequently

$$\int\limits_{\tilde{\Omega}_m \cap B_{m+1}^N(0)} |\nabla u_k|^2 \leqslant \int\limits_{\Omega_{k-1}} |\nabla u_k|^2 \leqslant \int\limits_{\Omega_k} \xi_k |\nabla u_k|^2 \to 0 \quad \text{as } k \to \infty.$$

Observe that

$$\begin{aligned} I(w_k) - I(u_k) = {} & \frac{1}{2} \int\limits_{\Omega} \big[\big(\xi^2 - 1\big)\big(|\nabla u_k|^2 + a(x)u_k^2\big) + |\nabla \xi|^2 |u_k|^2 + 2\xi u_k \nabla \xi \cdot \nabla u_k\big] \\ & - \int\limits_{\Omega} \big(F(w_k) - F(u_k)\big). \end{aligned}$$

Thus (3.18) follows from several estimates which are similar to the above. Hence $c \in \Theta(\tilde{\Omega}_m)$, this is contrary to the hypothesis, so there exists a positive solution u of problem (1.1). $\square$

4. The (PS)-conditions

Let Λ_2 be a smooth domain in $\mathbb{R}^N$ and Λ_1 a closed subset of Λ_2, we have the relation between $\alpha_M(\Lambda_2)$ and $\alpha_M(\Lambda_1)$.

Theorem 4.1. *Let $\Lambda_1 \subset \Lambda_2$. If I satisfies the $(PS)_{\alpha_M(\Lambda_1)}$-condition, then $\alpha_M(\Lambda_2) < \alpha_M(\Lambda_1)$.*

Proof. $\Lambda_1 \subset \Lambda_2$, so $\alpha_M(\Lambda_2) \leqslant \alpha_M(\Lambda_1)$. Suppose $\alpha_M(\Lambda_2) = \alpha_M(\Lambda_1)$. As a consequence of Ekeland's variational principle, there exists a sequence $\{u_k\} \subset M(\Lambda_1)$ such that $I(u_k) \to \alpha_M(\Lambda_1)$, $I'(u_k) \to 0$ in $H^{-1}(\Lambda_1)$. Since I satisfies the $(PS)_{\alpha_M(\Lambda_1)}$-condition, there exist a subsequence $\{u_k\}$, and $u \in H(\Lambda_1)$, satisfying $u_k \to u$ strongly in $H(\Lambda_1)$. Hence $I(u) = \alpha_M(\Lambda_1)$, $I'(u) = 0$. $I(u) = \alpha_M(\Lambda_2) = \inf_{u \in M(\Lambda_2)} I(u)$, it is known that every minimizer of $\alpha_M(\Lambda_2)$ is a critical point of I, therefore u solves problem (1.1) in Λ_2. By Lemma 3.1, $u > 0$ in Λ_2. This contradicts to $u \in H(\Lambda_1)$. Therefore $\alpha_M(\Lambda_2) < \alpha_M(\Lambda_1)$. $\square$

Theorem 4.2. *If* (f1)–(f4) *hold, then I satisfies the $(PS)_{\alpha_M(\Omega)}$-condition if and only if $\alpha_M(\Omega) < \lim_{k\to\infty} \alpha_M(\tilde{\Omega}_k)$.*

Proof. We first prove the sufficiency. Suppose $\alpha_M(\Omega) < \lim_{k\to\infty}\alpha_M(\tilde{\Omega}_k)$, then $\alpha_M(\Omega) < \alpha_M(\tilde{\Omega}_k)$ for some large $k\in\mathbb{N}$. Let $\{u_k\}$ be a $(PS)_{\alpha_M(\Omega)}$-sequence satisfying $u_k \rightharpoonup u$ weakly in $H(\Omega)$. By Lemma 3.2, $u\not\equiv 0$, then by Lemma 3.1, $u_k\to u$ strongly in $H(\Omega)$. We conclude that I satisfies the $(PS)_{\alpha_M(\Omega)}$-condition.

To prove the necessity, we argue indirectly. Suppose $\alpha_M(\Omega) = \lim_{k\to\infty}\alpha_M(\tilde{\Omega}_k)$, then $\alpha_M(\Omega) = \alpha_M(\tilde{\Omega}_k)$ for all $k\in\mathbb{N}$. We claim that I does not satisfy the $(PS)_{\alpha_M(\Omega)}$-condition in Ω. In fact, suppose on the contrary, I satisfies the $(PS)_{\alpha_M(\Omega)}$-condition in Ω. Then we claim that $I|_{H(\tilde{\Omega}_k)}$ satisfies the $(PS)_{\alpha_M(\Omega)}$-condition in $\tilde{\Omega}_k$ for all $k\in\mathbb{N}$. In fact, let $\{u_n\}\subset H(\tilde{\Omega}_k)\subset H(\Omega)$ satisfy $I(u_n)\to\alpha_M(\tilde{\Omega}_k)$, $I'(u_n)\to 0$ in $H^{-1}(\tilde{\Omega}_k)$. Since I satisfies the $(PS)_{\alpha_M(\Omega)}$-condition in Ω, there exist a subsequence $\{u_n\}$, and $u\in H(\Omega)$ satisfying $u_n\to u$ strongly in $H(\Omega)$; that is to say, $u_k\to u$ strongly in $H(\tilde{\Omega}_k)$. Therefore $I|_{H(\tilde{\Omega}_k)}$ satisfies the $(PS)_{\alpha_M(\tilde{\Omega}_k)}$-condition. By Theorem 4.1, $\alpha_M(\Omega) < \alpha_M(\tilde{\Omega}_k)$. This is a contradiction. □

5. Example: The strip domain with a hole

In this section, we study the domain is the strip domain with a hole, for simplicity of the presentation, we consider the case where f and a do not depend on x, so the problem is as follows:

$$\begin{cases} -\Delta u + au = f(u) \quad \text{in } \Sigma, \\ u > 0 \quad \text{in } \Sigma, \\ u \in H_0^1(\Sigma), \end{cases} \tag{5.1}$$

where $\Sigma = A^r\backslash D$, $D\subset B_\rho^N(0)\subset A^r$. Let

$$Q_s(\eta_0) = \{(\xi,\eta)\in\mathbb{R}^{N-1}\times\mathbb{R} \mid |\xi|<r,\ |\eta-\eta_0|<s\};$$

$$\tilde{Q}_s(\eta_0) = A^r\backslash\overline{Q_s(\eta_0)};$$

$$A_a^r = \{(\xi,\eta)\in A^r \mid \eta > a\}, \quad \text{where } a\in\mathbb{R};$$

$$S_a^r = \{(\xi,\eta)\in A^r \mid \eta < a\}, \quad \text{where } a\in\mathbb{R};$$

$$\Sigma_m = \Sigma\cap Q_m(0);$$

$$\tilde{\Sigma}_m = \Sigma\backslash\overline{Q_m(0)}.$$

Theorem 5.1.

(1) $\alpha_M(\Sigma) = \alpha_M(A^r)$;
(2) *I does not satisfy the $(PS)_{\alpha_M(\Sigma)}$-condition, and the only possible solutions of problem (5.1) in Σ are positive higher energy solutions.*

Proof. (1) Let $w \in H_0^1(A^r)$ be the positive solution of problem (5.1) in A^r with $I(w) = \alpha_M(A^r)$. Take $\{(0, \eta_n)\} \subset \Sigma$, $r_n \to \infty$ such that $Q_{r_n}(\eta_n) \subset \Sigma$. Consider the cut-out function $\psi \in C_c^\infty([0,\infty))$ such that

$$0 \leqslant \psi \leqslant 1, \quad \psi(t) = \begin{cases} 1 & \text{for } t \in [0,1], \\ 0 & \text{for } t \in [2,\infty). \end{cases}$$

Let

$$w_n(\xi,\eta) = \psi\left(\frac{2|\eta-\eta_n|}{r_n}\right) w(\xi, \eta-\eta_n).$$

Then

$$w_n \in H_0^1(\Sigma).$$

Since

$$\begin{aligned}
&\big\| w_n(\xi,\eta) - w(\xi,\eta-\eta_n)\big\|^2_{H^1(A^r)} \\
&\quad = \left\| \psi\left(\frac{2|\eta-\eta_n|}{r_n}\right) w(\xi,\eta-\eta_n) - w(\xi,\eta-\eta_n)\right\|^2_{H^1(A^r)} \\
&\quad \leqslant \int\limits_{A^r \cap \tilde{Q}_{r_n/2}(\eta_n)} \left(\left|\nabla w(\xi,\eta-\eta_n)\right|^2 + w(\xi,\eta-\eta_n)^2\right) + o(1) \\
&\quad = o(1),
\end{aligned}$$

and by (f2),

$$\begin{aligned}
&\int\limits_{A^r} \left|F\big(w_n(\xi,\eta)\big) - F\big(w(\xi,\eta-\eta_n)\big)\right| \\
&\quad = \int\limits_{A^r \cap \tilde{Q}_{r_n/2}(\eta_n)} \left|f\big(t w_n(\xi,\eta) + (1-t) w(\xi,\eta-\eta_n)\big)\right| \left|w_n(\xi,\eta) - w(\xi,\eta-\eta_n)\right| \\
&\quad \leqslant c \int\limits_{A^r \cap \tilde{Q}_{r_n/2}(\eta_n)} \left(\left|w(\xi,\eta-\eta_n)\right|^2 + \left|w(\xi,\eta-\eta_n)\right|^{p+1}\right) \\
&\quad = o(1), \quad \text{for some } 0 < t < 1, \\
&\int\limits_{A^r} \left|f\big(w_n(\xi,\eta)\big) w_n(\xi,\eta) - f\big(w(\xi,\eta-\eta_n)\big) w(\xi,\eta-\eta_n)\right| \\
&\quad \leqslant \int\limits_{A^r} \left|f\big(w_n(\xi,\eta)\big) w_n(\xi,\eta) - f\big(w_n(\xi,\eta)\big) w(\xi,\eta-\eta_n)\right| \\
&\qquad + \int\limits_{A^r} \left|f\big(w_n(\xi,\eta)\big) w(\xi,\eta-\eta_n) - f\big(w(\xi,\eta-\eta_n)\big) w(\xi,\eta-\eta_n)\right| \\
&\quad = o(1),
\end{aligned}$$

then we have

$$I(w_n)=\frac{1}{2}\|w_n\|^2_{H^1(\Sigma)}-\int\limits_{\Sigma}F(w_n)=I(w)+o(1)=\alpha_M\big(A^r\big)+o(1), \tag{5.2}$$

$$\begin{aligned}\big(I'(w_n),w_n\big)&=\|w_n\|^2_{H^1(\Sigma)}-\int\limits_{\Sigma}f(w_n)w_n=\|w\|^2_{H^1(A^r)}-\int\limits_{A^r}f(w)w+o(1)\\&=o(1).\end{aligned} \tag{5.3}$$

For $w_n\in H_0^1(\Sigma)$, $w_n\not\equiv 0$, by (5.2), (5.3), and Lemma 2.5, there exists $t_n\in(0,\infty)$ such that $t_nw_n\in M(\Sigma)$, $\{t_n\}$ is bounded, and $\alpha_M(\Sigma)\leqslant I(t_nw_n)\leqslant I(w_n)+o(1)=\alpha_M(A^r)+o(1)$. Hence we obtain $\alpha_M(\Sigma)=\alpha_M(A^r)$.

(2) By part (1) and Theorem 4.1, I does not satisfy the $(PS)_{\alpha_M(\Sigma)}$-condition. If u is a ground state solution of problem (5.1) in Σ, by putting $u=0$ in $A^r\setminus\Sigma$, we see that u could be regarded as an element of $H_0^1(A^r)$; then by the strong maximum principle, u would be a positive solution in A^r, a contradiction. Therefore the only possible solutions of problem (5.1) in Σ are positive higher energy solutions. □

With the same argument of the proof in Theorem 5.1, we have

Proposition 5.2. *Let Λ be a closed subset of A^r. If for any $s>0$ there exists $(0,\eta)\in\Lambda$ such that $Q_s(\eta)\subset\Lambda$, then $\alpha_M(\Lambda)=\alpha_M(A^r)$. As a more concrete example, Λ can be a upper half strip A_0^r, a strip domain with a hole Σ or the union of A_0^r with a bounded set.*

From Theorem 5.1 and Proposition 5.2, for any $\varepsilon_4>0$, there exists $z_-\in H_0^1(S^r_{-2\rho})\cap M(\Sigma)$ and $z_+\in H_0^1(A^r_{2\rho})\cap M(\Sigma)$ such that

$$\max\big(I(z_-),I(z_+)\big)<\alpha_M(\Sigma)+\varepsilon_4.$$

Set

$$\begin{aligned}&\Gamma=\big\{\gamma\in C\big([0,1],M(\Sigma)\big)\ \big|\ \gamma(0)=z_- \text{ and } \gamma(1)=z_+\big\},\\&\mu=\inf_{\gamma\in\Gamma}\max_{\theta\in[0,1]}I\big(\gamma(\theta)\big).\end{aligned}$$

We modified [3] to prove that there exists a $(PS)_\mu$-sequence with $\mu>\alpha_M(\Sigma)$, provided that z_+ and z_- are suitably chosen.

Denote that φ is a C^∞ function which satisfies $0\leqslant\varphi\leqslant 1$, $|\nabla\varphi|\leqslant 2/\rho$, $\varphi\equiv 0$ on $Q_\rho(0)$ and $\varphi\equiv 1$ on $\tilde{Q}_{2\rho}(0)$. Using the C^∞ function φ, from straightforward calculation, we have the following lemma.

Lemma 5.3. *For any $\varepsilon_5\in(0,\frac{\alpha_M(\Sigma)}{2})$, there exists $\delta=\delta(\varepsilon_5)>0$ such that if $u\in M(\Sigma)$ and $I(u)<\alpha_M(\Sigma)+\delta$ then $I(t_{\varphi u}\varphi u)<\alpha_M(\Sigma)+\varepsilon_5$, where $t_{\varphi u}>0$ such that $t_{\varphi u}\varphi u\in M(\Sigma)$.*

With Lemma 5.3, we want to show that $\mu>\alpha_M(\Sigma)$.

Theorem 5.4. *Let $\delta = \delta(\frac{\alpha_M(\Omega)}{4})$ be the number defined in Lemma* 5.3. *Choose $z_- \in H_0^1(S^r_{-2\rho}) \cap M(\Sigma)$ and $z_+ \in H_0^1(A^r_{2\rho}) \cap M(\Sigma)$ such that* $\max(I(z_-), I(z_+)) < \alpha_M(\Sigma) + \delta/4$. *Then $\mu \geqslant \alpha_M(\Sigma) + \delta$.*

Proof. Suppose $\mu < \alpha_M(\Sigma) + \delta$. From the definition of μ, there exists a $\gamma_0 \in \Gamma$ such that $\max_{\theta\in[0,1]} I(\gamma_0(\theta)) < \alpha_M(\Sigma) + \delta$. Let $\gamma(\theta) = t_{\varphi\gamma_0(\theta)}\varphi\gamma_0(\theta)$, it follows from Lemma 5.3 that $\gamma \in \Gamma$ and

$$\max_{\theta\in[0,1]} I\big(\gamma(\theta)\big) < \alpha_M(\Sigma) + \varepsilon_5 < \frac{3}{2}\alpha_M(\Sigma). \tag{5.4}$$

By the definition of φ, $\gamma(\theta) = \gamma_+(\theta) + \gamma_-(\theta)$, where $\gamma_+(\theta) \in H_0^1(A^r_{2\rho})$ and $\gamma_-(\theta) \in H_0^1(S^r_{-2\rho})$. We claim that

$$\text{there exists a } \theta_0 \in (0,1) \quad \text{such that} \quad \gamma_+(\theta_0) \in M(\Sigma) \quad \text{and} \quad \gamma_-(\theta_0) \in M(\Sigma). \tag{5.5}$$

Assuming (5.5) for now, we obtain $I(\gamma(\theta_0)) = I(\gamma_+(\theta_0)) + I(\gamma_-(\theta_0)) > \alpha_M(\Sigma) + \alpha_M(\Sigma) = 2\alpha_M(\Sigma)$, which contradicts (5.4).

It remains to show (5.5) to complete the proof. Since $\gamma_+(0) = 0$ and $\gamma_+(1) = z_+$, there exists a $\theta_1 \in (0,1)$ such that $I'(\gamma_+(\theta_1))\gamma_+(\theta_1) > 0$. This together with $\gamma(\theta_1) \in M(\Sigma)$ implies that $I'(\gamma_-(\theta_1))\gamma_-(\theta_1) < 0$. Let

$$\theta_2 = \sup\big\{\theta \;\big|\; I'\big(\gamma_-(\theta)\big)\gamma_-(\theta) < 0 \text{ or } \gamma_-(\theta) \in M(\Sigma)\big\}. \tag{5.6}$$

Since $\gamma_-(1) = 0$ and $\gamma_-(0) = z_-$, it follows that $\theta_2 \in (0,1)$. Using $I \in C^1$ and $I'(\gamma_-(\theta_2))\gamma_-(\theta_2) = 0$. Since $\gamma(\theta_2) \in M(\Sigma)$, it follows that $I'(\gamma_+(\theta_2))\gamma_+(\theta_2) = 0$.

To complete the proof of (5.5), we need to show that $\gamma_-(\theta_2) \neq 0$ and $\gamma_+(\theta_2) \neq 0$. We argue indirectly. If $\gamma_-(\theta_2) = 0$, then either $\gamma_-(\theta) = 0$ for all $\theta \in (\theta_2, 1)$ or there exists a $\theta_3 \in (\theta_2, 1)$ such that $I'(\gamma_-(\theta_3))\gamma_-(\theta_3) > 0$. This contradicts (5.6). Suppose $\gamma_+(\theta_2) = 0$. Then there exists a $\theta_4 \in (\theta_2, 1)$ such that $I'(\gamma_+(\theta_4))\gamma_+(\theta_4) > 0$. This together with $\gamma(\theta_4) \in M(\Sigma)$ yields $I'(\gamma_-(\theta_4))\gamma_-(\theta_4) < 0$, which again violates (5.6). Thus the proof is complete. □

Then we will show that the existence of a Palais–Smale sequence with the (PS)-value μ.

Theorem 5.5. *There exists a $(PS)_\mu$-sequence, where μ is the number defined above Lemma* 5.3.

Proof. Suppose there does not exist a $(PS)_\mu$-sequence. Then there exist $b > 0$ and $\hat{\varepsilon} > 0$ such that $\|I'(u)\| \geqslant b$ for all u with $\mu - \hat{\varepsilon} < I(u) \leqslant \mu + \hat{\varepsilon}$. We may assume without loss of generality that $b < 1$ and $\hat{\varepsilon} < \frac{1}{2}(\mu - \alpha_M(\Sigma) - \frac{\delta}{4})$, where δ is the number defined in Lemma 5.3.

Let $Y_1 = \{u \in M(\Sigma) \mid \|I'(u)\| \leqslant \frac{b}{2}$ and $I(u) \leqslant \frac{3\mu}{2}\}$ and $Y_2 = \{u \in M(\Sigma) \mid \|I'(u)\| \geqslant b$ and $I(u) \leqslant \frac{3\mu}{2}\}$. Choose

$$\varepsilon \in (0, \varepsilon_1), \quad \text{where } \varepsilon_1 = \min\bigg(\hat{\varepsilon}, \frac{b^2}{2}, \frac{b}{4}\bigg). \tag{5.7}$$

Let $Y_3 = \{u \in M(\Sigma) \mid I(u) \leqslant \mu - \hat{\varepsilon} \text{ or } I(u) \geqslant \mu + \hat{\varepsilon}\}$ and $Y_4 = \{u \in M(\Sigma) \mid \mu - \varepsilon \leqslant I(u) \leqslant \mu + \varepsilon\}$. For $u \in M(\Sigma)$, set $g_1(u) = \frac{\|u-Y_3\|}{\|u-Y_3\|+\|u-Y_4\|}$ and $g_2(u) = \frac{\|u-Y_1\|}{\|u-Y_1\|+\|u-Y_2\|}$. Let $X(u)$ be a pseudo-gradient vector field for I on $M(\Sigma)$ and

$$W(u) = -g_1(u)g_2(u)h\big(\|X(u)\|\big)X(u), \tag{5.8}$$

where $h(s) = 1$ if $s \in [0, 1]$ and $h(s) = 1/s$ if $s \geqslant 1$.

Consider the Cauchy problem:

$$\frac{d\eta}{dt} = W(\eta), \quad \eta(0, u) = u. \tag{5.9}$$

The basic existence-uniqueness theorem for ordinary differential equations implies that, for each $u \in M(\Sigma)$, (5.9) has a unique solution $\eta(t, u)$ which is defined for t in a maximal interval $[0, T(u))$. Moreover, since $\|W(u)\| \leqslant 1$ and $M(\Lambda)$ is a closed subset of $H_0^1(\Sigma)$, so $T(u) = +\infty$. Since

$$\frac{d}{dt}I\big(\eta(t,u)\big) = -I'\big(\eta(t,u)\big)g_1\big(\eta(t,u)\big)g_2\big(\eta(t,u)\big)h\big(\big\|X\big(\eta(t,u)\big)\big\|\big)X\big(\eta(t,u)\big).$$

Define $\tilde{I}^a = \{u \in M(\Sigma) \mid I(u) \leqslant a\}$, since $I(\eta(t, u))$ is a nonincreasing function of t, hence

$$\eta\big(1, \tilde{I}^{\mu-\varepsilon}\big) \subset \tilde{I}^{\mu-\varepsilon}. \tag{5.10}$$

We claim

$$\eta(1, Y_4) \subset \tilde{I}^{\mu-\varepsilon}. \tag{5.11}$$

Indeed, if there exists $u \in Y_4$ such that $\eta(1, u) \notin \tilde{I}^{\mu-\varepsilon}$, then, for all $t \in [0, 1]$, $\eta(t, u) \in Y_4$. Consequently $g_1(\eta(t, u)) = 1$ and $g_2(\eta(t, u)) = 1$. If for some $t \in (0, 1)$, $\|X(\eta(t, u))\| \leqslant 1$, then $h(\|X(\eta(t, u))\|) = 1$ and

$$\frac{d}{dt}I\big(\eta(t,u)\big) \leqslant -\big\|I'\big(\eta(t,u)\big)\big\|^2 \leqslant -b^2. \tag{5.12}$$

On the other hand, if for some $t \in (0, 1)$, $\|X(\eta(t, u))\| > 1$, then by the definition of pseudo-gradient vector field,

$$\frac{d}{dt}I\big(\eta(t,u)\big) \leqslant -\big\|I'\big(\eta(t,u)\big)\big\|^2\big\|X\big(\eta(t,u)\big)\big\|^{-1} \leqslant -\frac{1}{2}\big\|I'\big(\eta(t,u)\big)\big\| \leqslant -\frac{b}{2}. \tag{5.13}$$

Since $\eta(t, u) \in Y_4$ for all $t \in [0, 1]$, by (5.12) and (5.13), we have

$$2\varepsilon \geqslant I\big(\eta(0,u)\big) - I\big(\eta(1,u)\big) = -\int_0^1 \frac{d}{dt}I\big(\eta(t,u)\big)\,dt \geqslant \min\left(\frac{b}{2}, b^2\right). \tag{5.14}$$

Since (5.14) is contrary to (5.7), we conclude that (5.11) must hold. Combining (5.10) and (5.11), we have

$$\eta\big(1, \tilde{I}^{\mu+\varepsilon}\big) \subset \tilde{I}^{\mu-\varepsilon}. \tag{5.15}$$

By the definition of μ, there exists a $\gamma \in \Gamma$ such that $\max_{\theta\in[0,1]} I(\gamma(\theta)) < \mu + \varepsilon$. Let $\gamma_1(\theta) = \eta(1, \gamma(\theta))$. It follows from (5.15) that

$$\max_{\theta\in[0,1]} I\big(\gamma_1(\theta)\big) \leqslant \mu - \varepsilon. \tag{5.16}$$

Since $g_1(u) = 0$ if $u \in \tilde{I}^{\mu-\hat{\varepsilon}}$, it follows from (5.8) and (5.9) that $\eta(1, u) = u$ if $u \in \tilde{I}^{\mu-\hat{\varepsilon}}$. In particular, $\max(I(z_+), I(z_-)) < \alpha_M(\Sigma) + \frac{\delta}{4}$ implies $\gamma_1(0) = \gamma(0)$, $\gamma_1(1) = \gamma(1)$ and consequently $\gamma_1 \in \Gamma$. But then (5.16) is contrary to the definition of μ. The proof is complete. □

We are now ready to prove the existence of a positive higher energy solution of problem (5.1).

Theorem 5.6. *Assume that* (f1)–(f4) *hold. If* $\mu \notin \Theta(\tilde{\Sigma}_m)$ *for some* $m \in \mathbb{N}$*, then there exists a higher energy solution of problem* (5.1) *in* Σ.

Proof. By Theorems 5.4 and 5.5, there exists a $(PS)_\mu$-sequence with $\mu > \alpha_M(\Sigma)$, then by Theorem 3.5, we obtain a positive higher energy solution of problem (5.1) in Σ. □

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